

On Throughput Scaling of Wireless Networks: Effect of Node Density and Propagation Model

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Oct 18, 2006

Abstract

This paper derives a lower bound to the per-node throughput achievable by a wireless network when n source-destination pairs are randomly distributed throughout a disk of radius n^γ , $\gamma \geq 0$, propagation is modeled by attenuation of the form $1/(1+d)^\alpha$, $\alpha > 2$, and successful transmission occurs at a fixed rate W when received signal to noise and interference ratio is greater than some threshold β , and at rate 0 otherwise. The lower bound has the form $n^{1-\gamma}$ when $\gamma < 1/2$, and $(n \ln n)^{-1/2}$ when $\gamma \geq 1/2$. The methods are similar to, but somewhat simpler than, those in the seminal paper by Gupta and Kumar.

¹This work was supported by NSF grants CCR-0329715 and ANI-0238035. Portions of this work were presented at the IEEE International Symposium on Information Theory, Seattle, July 2006.

1 Introduction

The pioneering work of Gupta and Kumar [1] has led to many studies of scaling laws for the asymptotically achievable throughput in wireless networks under a variety of network models and assumptions. Such scaling laws help us understand the fundamental performance limits of these networks and how efficiency changes as network conditions change. Some examples include [2] where the nodes are allowed to move; [3, 4, 5], where many-to-one type of communications is considered; [6, 5], where cooperative communication schemes are employed to improve network throughput; and [7], where scaling laws are derived using directional antenna assumptions. Other examples can be found in the June 2006 Special Issue on Networking and Information Theory of the *IEEE Trans. Inform. Theory*.

All of these scaling results are highly dependent on the various assumptions made, such as on the network topology (e.g., planar, linear, ring, sphere, etc.), the purpose of the network (e.g., many-to-many vs. many-to-one communications), the physical layer models (e.g., different signal propagation and interference models), and the asymptotic density of nodes (e.g. increasing to infinity or remaining constant).

This paper focuses on two such aspects – the underlying model for signal propagation and the asymptotic density of nodes. We focus on the many-to-many communications task. Specifically, a set of n nodes are randomly distributed over some region A , and each node randomly chooses another node to whom to transmit data. All such transmissions use the same power P , which the designer can choose, and communicate bits at some fixed rate W that does not depend on P . Transmissions are received in the presence of interference from other nodes transmitting at the same time, as well as from background noise. They are modeled as successful if the signal to interference and noise ratio (SINR) at the receiver is above some threshold and unsuccessful if not. For this task, [1] found the maximum attainable throughput per node is² $\Omega(\frac{1}{\sqrt{n \ln n}})$ and $\mathcal{O}(\frac{1}{\sqrt{n}})$ bits/sec, assuming a propagation law in which received power decays as $\frac{1}{d^\alpha}$ with transmission distance d , for some $\alpha > 2$. That is, it scales at least as $\frac{1}{\sqrt{n \ln n}}$, but no larger than $\frac{1}{\sqrt{n}}$.

While [1] assumed that the n nodes were randomly distributed over a fixed region A , and consequently, the network becomes denser as n increases, the maximum throughput is actually independent of the size of the region. For example, it does not change if the region size scales with n , as we will wish to consider in this paper. To see this, consider a specific set of nodes transmitting simultaneously in some region A , each with power P , and each to its own receiving node. Now suppose the positions of all transmitting and receiving nodes are scaled by a factor μ , and the transmit power is scaled by the factor μ^α . Then the SINR (to be defined in Section 3 in the obvious way) will be the same at each of the scaled receiving nodes, as it was at each of the original unscaled nodes. It follows that the maximum attainable throughput is not affected by a scaling of the region over which the nodes are distributed. For example, the $\Omega(\frac{1}{\sqrt{n \ln n}})$ throughput law applies equally when the region A is fixed and the density of nodes increases linearly with n , or when the density of nodes is fixed and area of A increases linearly with n .

In contrast, Arpacioğlu and Haas [8] have shown that when the propagation model has the form $\frac{1}{(1+d)^\alpha}$, for some $\alpha > 2$, and the region A remains fixed (so node density increases linearly with

²We use the notation $\mathcal{O}(f_n)$, $\Omega(f_n)$ and $\Theta(f_n)$, in the conventional way, i.e., to characterize a quantity x_n depending on n for which there are finite constants $c > 0$, $d > 0$ and n_0 such that, for all $n > n_0$, respectively, $x_n < df_n$, $x_n > cf_n$, and $cf_n < x_n < df_n$.

n), the maximum attainable throughput decreases dramatically to $\Theta(\frac{1}{n})$, which is the throughput attained by simple time sharing among the n nodes. On the other hand, the two propagation models are essentially equivalent in the far field. As a result, if the area of A increases at least linearly with n , then because the distances between nearest nodes are not decreasing with n to zero, it is relatively easy to see that the maximum throughput is the same for both models, i.e., it is $\Omega(\frac{1}{\sqrt{n \ln n}})$ and $\mathcal{O}(\frac{1}{\sqrt{n}})$. The above cited results on the maximum attainable throughputs are summarized in Table 1. One concludes that throughput depends significantly on the assumptions about propagation model and node density.

	Propagation Models	
	$\frac{1}{d^\alpha}$	$\frac{1}{(1+d)^\alpha}$
fixed area ($\gamma = 0$)	$\Omega(\frac{1}{\sqrt{n \ln n}}), \mathcal{O}(\frac{1}{\sqrt{n}})$ [1]	$\Theta(\frac{1}{n})$ [8]
fixed density ($\gamma = \frac{1}{2}$)	$\Omega(\frac{1}{\sqrt{n \ln n}}), \mathcal{O}(\frac{1}{\sqrt{n}})$	$\Omega(\frac{1}{\sqrt{n \ln n}}), \mathcal{O}(\frac{1}{\sqrt{n}})$

Table 1: Throughput scaling results for random networks under different propagation models and network density assumptions.

In this paper, we focus on the $\frac{1}{(1+d)^\alpha}$ propagation model and the gap, evident in the rightmost column of Table 1, between the maximum throughputs attainable for fixed area and fixed density. Specifically, we ask how attainable throughput changes as the node deployment scenario ranges from fixed area to fixed or decreasing density. We do this by considering the network region A to be a disk with radius n^γ , where $\gamma \geq 0$ is a parameter that determines the deployment scenario. The choice $\gamma = 0$ corresponds to a network with fixed area and node density increasing linearly with n . The choice $\gamma = \frac{1}{2}$ corresponds to a network with area increasing linearly with n and density remaining constant. Intermediate values of γ correspond to the network density increasing sublinearly, while $\gamma > \frac{1}{2}$ corresponds to decreasing network density. We consider time-slotted systems and measure throughput in bits/slot, which of course can be easily converted to bit/sec. The principal result of the paper is that throughput $\Omega(\frac{1}{n^{1-\gamma}})$ is attainable when $\gamma < \frac{1}{2}$, whereas throughput $\Omega(\frac{1}{\sqrt{n \ln n}})$ is attainable when $\gamma \geq \frac{1}{2}$. If it is desired to measure throughput in bit-meters/slot, then these results are multiplied by n^γ .

For $\gamma = 0$, the attainable throughput $\Omega(\frac{1}{n^{1-\gamma}})$ is consistent with the $\Theta(\frac{1}{n})$ result found in [8]. As γ increases towards $\frac{1}{2}$, the attainable throughput $\Omega(\frac{n^\gamma}{n})$ increases, due essentially to the fact that as γ increases, there is room for more simultaneous transmitters. For $\gamma \geq \frac{1}{2}$, the attainable throughput scaling rate saturates at $\Omega(\frac{1}{\sqrt{n \ln n}})$. This is the rate found for $\gamma = 0$ and the $\frac{1}{d^\alpha}$ propagation model [1], that also applies to $\gamma = \frac{1}{2}$ and the $\frac{1}{(1+d)^\alpha}$ propagation model (see Table 1). For $\gamma = \frac{1}{2} - \epsilon$ and very small ϵ , one might be tempted to interpret the result as saying that throughput $\Omega(\frac{1}{n^{1-\gamma}}) \approx \Omega(\frac{1}{\sqrt{n}})$ is attainable, which would be larger than the attainable throughput $\Omega(\frac{1}{\sqrt{n \ln n}})$ for $\gamma = \frac{1}{2}$, and would contradict the notion that attainable throughput does not decrease when γ increases. However, the result actually says the attainable throughput is $\Omega(\frac{1}{\sqrt{n n^\epsilon}})$, which is a smaller lower bound than $\Omega(\frac{1}{\sqrt{n \ln n}})$, the attainable throughput for $\gamma = \frac{1}{2}$, no matter how small ϵ is.

Interestingly, Franceschetti et al. [9] have shown recently that larger throughput, $\Omega(\frac{1}{\sqrt{n}})$, is attainable in a variety of situations. These include the $\frac{1}{d^\alpha}$ propagation model and both a fixed area ($\gamma = 0$) and a fixed density ($\gamma = \frac{1}{2}$) network region. They also include a fixed density network and a propagation model that is bounded, like the $\frac{1}{(1+d)^\alpha}$ propagation model considered in the present paper. For the $\frac{1}{d^\alpha}$ propagation model, the previously mentioned invariance of SINR to dimension scaling of the network region and appropriate scaling of power implies that throughput $\Omega(\frac{1}{\sqrt{n}})$ is in fact attainable for all $\gamma \geq 0$. For the bounded propagation model, it is not evident what happens when $\gamma < \frac{1}{2}$. The larger throughputs demonstrated in [9] are obtained assuming that the rate of successful transmission between two nodes equals the capacity of an additive Gaussian channel with signal to noise ratio equal to the received SINR. This contrasts with the two-rate transmission assumed in [1, 8] and the present paper, in which the rate is W when received SINR exceeds a threshold and 0 otherwise. The construction in [9] also adopted a hierarchical structure where packets are first sent to a backbone from which they are routed to the destination. This contrasts with the straight line shortest path type of routing used in [1, 8] and the present paper.

Assuming the $\frac{1}{(1+d)^\alpha}$ propagation law, $\gamma \geq 0$, and the two-rate transmission model, it may well be that throughput cannot scale at rates above those we show to be attainable. However, no such proof or claim is offered in this paper.

In the remainder of the paper, Section 2 introduces the many-to-many communication task, along with a concrete specification of a system for this task, its throughput and the notion of a successful system. The latter is determined by a propagation model and a criterion for judging the success of a transmission in the presence of interfering transmitters and background noise. The specific success criterion and propagation model used in this paper are introduced in Section 3. Section 4 introduces distance-based success criteria, which are like the protocol models used in [1], and it discusses their relationship to the SINR-based physical model of [1]. Section 5 states and proves the main result. Section 6 summarizes and makes concluding remarks. Finally, a few details are relegated to appendices.

While the methods used here are related to those used in previous work (e.g., the use of distance-based protocol models for determining when a set of simultaneous transmitters will not interfere with each other [1], and the use of straight lines intersecting cells of a partition to determine routes), they differ in key respects (e.g., the dividing of the load as equally as possible among the nodes within a partition cell, and the use of the Chernoff bound instead of uniform convergence of the weak law of large numbers). As one benefit, the new methods permit straightforward analysis of throughput scaling on a disk, rather than the surface of a sphere, despite the hot-spot-at-the-center problem. In addition, we clarify the role of protocol models, and their relations to physical models, in aiding the design of a system and the demonstration of attainable throughputs. To illustrate the generality of our methods, we also indicate in Section 5 how they can straightforwardly demonstrate the original $\Omega(\frac{1}{\sqrt{n \ln n}})$ throughput result of [1].

2 The Many-to-Many Communication Task

A set of n nodes, $\Sigma_n = \{s_1, \dots, s_n\}$, is distributed over a disk $A_n \subset \mathcal{R}^2$ with radius n^γ , called the *network region*, where $s_i \in A_n$ denotes the location within the disk of the i th node, where $\gamma \geq 0$ is a fixed parameter that characterizes how the area of the disk and the density of nodes scale with

n . Each node serves as a source of bits that it wishes to communicate to some destination. For each source s_i , another of the n nodes, denoted d_i , is designated as the *destination* for its bits. As a result, there is a *source-destination set* $\mathcal{P}_n = \{(s_1, d_1), \dots, (s_n, d_n)\}$ consisting of n source-destination pairs, each representing a desired *conversation*. Note that a node may serve as the destination for more than one source.

Each of the n sources has an infinite number of bits it wishes to communicate to its destination node, as quickly as possible. Communication uses simple multihop relaying with a time slotted system. We make the usual assumption that the source-destination set is random. Specifically, s_1, \dots, s_n are drawn independently, each with a uniform distribution on the disk. Then for each s_i , the destination d_i is equally likely to be any $s_j, j \neq i$, independent of all other s 's and d 's. (Two sources may have the same destination.)

Roughly speaking, for a given n , one wishes to find the largest number λ such that for all source-destination sets \mathcal{P}_n , except a set with small probability, λ bits/slot can be successfully transmitted from each source to its destination. In this paper we do not claim to have found the largest possible λ . However, we are able to show that with $\lambda_n = \Omega(\frac{1}{n^{1-\gamma}})$, $\gamma < \frac{1}{2}$ and $\lambda_n = \Omega(\frac{1}{\sqrt{n \ln n}})$, $\gamma \geq \frac{1}{2}$, with probability approaching one as $n \rightarrow \infty$, there exist systems that send λ_n bits/slot.

2.1 System Definition

We now describe the kind of system to be used for the many-to-many task. This is basically an explicit formalization of the kind of system that appears in prior work. There is a transmitter and receiver at each of the n nodes. The antennas at each node are omnidirectional. All transmitters use the same power P , which we get to choose and which may depend on n and the specific source-destination set \mathcal{P}_n . (However, we will see in Theorem 5 that when our system is optimized and $\gamma < \frac{1}{2}$, the power P can remain constant.) As mentioned earlier, transmissions occur in slots. We assume there is a fixed $W > 0$ such that each transmitter can transmit at most one packet, consisting of W bits, in one slot, regardless of P , n or any other factors. Such transmissions are received throughout the network region A_n in the presence of background noise with power N_o and interference from other transmitters transmitting at the same time. As a result, the packets might or might not be successfully received by an intended receiver. Criteria for determining success will be introduced later.

Each receiver can store an arbitrary number of packets, for later retransmission. However, we will see there is little need for such storage.

To communicate bits from the sources to their destinations, each source-destination pair needs a *route* and a *schedule*. A *route* for source-destination pair (s_i, d_i) is a finite sequence of *hops*, $h_i = (h_{i,1}, \dots, h_{i,J_i})$, from s_i to d_i with the j th hop of the route being a pair $h_{i,j} = (t_{i,j}, r_{i,j})$ indicating that node $t_{i,j} \in \Sigma_n$ is to transmit bits originating at s_i with the intention that they be received³ by node $r_{i,j} \in \Sigma_n$. The first hop has the form $h_{i,1} = (s_i, r_{i,1})$, subsequent hops have $r_{i,j} = t_{i,j+1}$, and the last hop has the form $h_{i,J_i} = (t_{i,J_i}, d_i)$. Since nodes are presumed to be able to indefinitely store packets received in previous time slots, there is no need to allow routes to have loops, i.e. for one node to appear twice in a route. Accordingly we disallow loops, which implies

³We say ‘‘intention’’ because the omnidirectionality of the antennas means that other nodes will also hear the transmission, and because noise or interference may prevent the transmission from being successfully received.

that all routes have length $n - 1$ or less. Paths for different source-destination pairs may have different numbers of hops. The *length* of a hop $h = (t, r)$ is the Euclidean distance $\|t - r\|$.

A *schedule* for route $h_i = (h_{i,1}, \dots, h_{i,J_i})$ is a sequence of positive integers $\sigma_i = (\sigma_{i,1}, \dots, \sigma_{i,J_i})$ assigning a time slot to each hop of the route. Specifically, node $t_{i,j}$ makes its transmission of hop $h_{i,j}$ in time slot $\sigma_{i,j}$. Combining the notions of route and schedule, each source-destination pair (s_i, d_i) is assigned a *scheduled route* $H_i = ((h_{i,1}, \sigma_{i,1}), \dots, (h_{i,J_i}, \sigma_{i,J_i}))$. The hops in a route need not be assigned slots in increasing order; i.e., we permit $\sigma_{i,j} > \sigma_{i,j+1}$.

We now define a *system* \mathcal{S}_n for source-destination set \mathcal{P}_n to be a set of n scheduled routes $\{H_1, \dots, H_n\}$. Such a system is assumed to operate periodically with period $p = \max_{i,j} \sigma_{i,j}$, which is the largest slot assignment of any hop of any route. That is, in steady state, the j th hop of route $h_i = (h_{i,1}, \dots, h_{i,J_i})$ is transmitted in slot σ_j of each epoch of p slots. The reason for restarting each route synchronously at the beginning of each epoch will be explained shortly.

We also require scheduled routes of a system to be *compatible* in the sense that no two hops, either from the same or different routes, can be scheduled to require transmission from the same node in the same slot. This requirement stems from our assumption that a node can transmit at most once within a slot. The previously stated assumption that all routes are transmitted again in every epoch of length p (instead of, say, each route cycling asynchronously) is designed to permit compatibility to be checked straightforwardly.

In summary, a system $\mathcal{S}_n = \{H_1, \dots, H_n\}$ for a set of source-destination pairs \mathcal{P}_n consists of a compatible set of n scheduled routes, one for each source-destination pair in \mathcal{P}_n , and with the latest time slot assigned to any hop being defined as the *period* p of the system. For future use, for $j \in \{1, \dots, p\}$, let us define the *hop set* \mathcal{H}_j to be the set of hops (t, r) that the system specifies as transmitting in the j th slot. That is, \mathcal{H}_j contains a hop $h = (t, r)$ if h is a hop in some scheduled route that is scheduled for the j th time slot. Let us also define the *transmission set* \mathcal{T}_j to be the set of nodes that the system specifies as scheduled for the j th slot. Note that for any \mathcal{P}_n , there obviously exists a set of routes, and for any set of routes, one can always find a set of compatible schedules for these routes. For example, although not very efficient, one could define a schedule in which each hop of each route is assigned to a distinct slot. Therefore, there always exists a system for any \mathcal{P}_n .

We now describe concretely how a system $\mathcal{S}_n = \{H_1, \dots, H_n\}$ with period p for source-destination set $\mathcal{P}_n = \{(s_1, d_1), \dots, (s_n, d_n)\}$ transmits data from the sources to the destinations. For each $i \in \{1, \dots, n\}$, the first packet from s_i is transmitted via hop $h_{i,1}$ in slot $\sigma_{i,1}$ of the first epoch of p slots. Then, if $\sigma_{i,2} > \sigma_{i,1}$, this packet is relayed via the second hop $h_{i,2}$, also in the first epoch. If not, it is transmitted in the second epoch. Subsequently, for each $j > 2$, the packet is relayed via hop $h_{i,j} = (t_{i,j}, r_{i,j})$ in slot $\sigma_{i,j}$ of the first epoch in which the packet has been received at $t_{i,j}$ prior to slot $\sigma_{i,j}$. Moreover, transmission of subsequent packets from s_i to d_i are pipelined so that in steady-state, within each epoch, one new packet is generated by s_i , each hop $h_{i,j}$ of the corresponding scheduled route is executed once, and one packet from s_i (typically generated in an earlier epoch) is received at destination d_i . And this happens for each source-destination pair. Compatibility ensures that no node is asked to make two transmissions in one slot.

Notice that, as assumed earlier, there is no need for the $\sigma_{i,j}$'s of a scheduled route to be in increasing order. A node simply stores each received packet until it is time to transmit it, either in the present or next epoch. As a result, it will never store more than one packet at a time from

one source-destination pair.

2.2 Success Criteria

We assume the existence of a *transmission success criterion* that determines whether or not a given transmission will be successful. Specifically, when a node at location⁴ t transmits to a node at location r in the presence of background noise with power N_o and simultaneous transmissions from locations in the set $T = \{t_1, t_2, \dots, t_{M-1}\}$, the criterion determines whether this communication is successful. Such a criterion can be characterized by a *success indicator function* $\psi(t, r, T, P, N_o)$ of the form $\psi : \mathbb{R}^2 \times \mathbb{R}^2 \times \overline{\mathbb{R}^2} \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \{0, 1\}$, where $\overline{\mathbb{R}^2}$ denotes the set of all finite subsets of \mathbb{R}^2 , \mathbb{R}_+ indicates the set of nonnegative real numbers, and $\psi(t, r, T, P, N_o) = 1$ indicates that conditions are suitable for a successful transmission from t to r in the presence of background noise with power N_o and simultaneous transmissions from the locations in T , whereas $\psi(t, r, T, P, N_o) = 0$ indicates they are not.

Although a success criterion in this general form can be used to model communication in a variety of networks, since we are dealing with wireless networks, we will use a success indicator criterion that is characterized by a propagation model η and a power indicator function ϕ . A *propagation model* is a function $\eta : [0, \infty) \rightarrow [0, \infty)$ such that $\eta(d)$ determines the fraction of transmit power that is received at distance d from the transmitter. A *power indicator function* is a binary function $\phi(P', \{P_1, P_2, \dots, P_{M-1}\}, N_o)$ that equals one if the power P' received at r from the transmitter at t , the set of powers $\{P_1, P_2, \dots, P_{M-1}\}$ received at r from the other transmitters at locations in T , and the background noise level N_o are such that the transmission from t to r is successful, and zero if not. We now restrict attention to success indicator functions of the form

$$\psi(t, r, T, P, N_o) = \phi(P\eta(\|t - r\|), \{P\eta(\|t_1 - r\|), \dots, P\eta(\|t_{M-1} - r\|)\}, N_o) .$$

With a transmission success criterion in hand, one may now define a hop set \mathcal{H} to be successful if for every hop $(t, r) \in \mathcal{H}$, transmission from t to r is successful in the presence of transmissions from all other transmitters in the transmission set T corresponding to \mathcal{H} . Next, one may define a system to be successful if all of its hop sets are successful. Note that in the situation described above in which a propagation model η is available, the success of a hop set or a system will depend on P , N_o and the propagation model η , as well as the locations of the transmitters and receivers of the hops in the hop set.

2.3 Throughput

If a system \mathcal{S}_n with period p for source-destination set \mathcal{P}_n is successful, i.e., if all hop transmissions are received successfully, then in steady-state the system delivers one packet, consisting of W bits, from each source s_i to its destination d_i in each epoch of p slots. Accordingly, we define the *throughput* for a system to be

$$\lambda = \frac{W}{p} .$$

Clearly, throughput is a meaningful quantity only when the system is successful. To attain large throughput, one needs to design a set of compatible scheduled routes with p as small as possible.

⁴Whereas s_i denotes the location of the i th node, letters t and r with one or no subscripts, such as t_i and r_i , are used as variables to denote the location of some transmitter and receiver, respectively.

It would have been possible to permit more than one route for each source-destination pair. In this case, one would define throughput to be mW/p , where m is the minimum number of routes per source-destination pair. However, since this paper focuses only on finding a lower bound to achievable throughput, there is no need to consider such.

Note that the order in which hops in a route are scheduled has no effect on throughput, though it will effect the delay until the first packets from each source appear at their destinations.

The problem of interest is to learn the order of the maximum possible throughput that is attainable with high probability when n is large, where probability refers to the randomness in the source-destination set. Our main result is: for the $\frac{1}{(1+d^\alpha)}$ propagation model, there exist constants $0 < c, \bar{c} < \infty$ such that for each n there is a $P_n > 0$ and for each source-destination set \mathcal{P}_n , there is a system $\mathcal{S}_n(\mathcal{P}_n)$ with power P_n and throughput denoted $\lambda_n(\mathcal{P}_n)$ such that $\Pr(\mathcal{S}_n(\mathcal{P}_n) \text{ is successful}) \rightarrow 1$. In addition, if $\gamma < \frac{1}{2}$, $\Pr(\lambda_n(\mathcal{P}_n) \geq \frac{W}{cn^{1-\gamma}}) \rightarrow 1$, while if $\gamma \geq \frac{1}{2}$, $\Pr(\lambda_n(\mathcal{P}_n) \geq \frac{W}{\bar{c}\sqrt{n \ln n}}) \rightarrow 1$. Note that the success and throughput of the system $\mathcal{S}_n(\mathcal{P}_n)$ depend on the locations of the source-destination pairs in \mathcal{P}_n .

3 Propagation Model and Success Criterion Choices

In this section we indicate the specific propagation models and success criterion that we use in this paper.

3.1 Propagation Models

In most of the prior work, e.g. [1], the following signal propagation model is adopted:

Definition 1 - Propagation Model A:

$$\eta(d) = \frac{1}{d^\alpha},$$

where $\alpha > 0$ is a constant whose value depends on the conditions of the channel.

Notice that under Model A, when nodes become very close, as happens for example when n increases and $\gamma = 0$ so that the network region A_n remains fixed, the received power will be larger than the transmitted, which is not reasonable. In other words, Model A makes sense only as a far field assumption. This was noted by Arpacioglu and Haas in [8] and by Dousse and Thiran in [10]. In particular, [8] considered the following alternative model:

Definition 2 - Propagation Model B:

$$\eta(d) = \frac{1}{(1+d)^\alpha}.$$

With this model, no matter how close two nodes become, the received power is upper bounded by the transmit power. Similarly, [10] considered a broad class of decreasing propagation models that are upper bounded as d approaches 0.

The difference between models A and B has an important implication. Consider a node t transmitting to node r and some other transmitting node t' whose power received at r appears as noise.

Under Model A, the ratio of powers received from each is $\frac{\eta(\|t-r\|)}{\eta(\|t'-r\|)} = \left(\frac{\|t'-r\|}{\|t-r\|}\right)^\alpha$. As $\|t-r\|$ decreases, as long as $\|t'-r\|$ does not decrease as fast, the ratio of $\eta(\|t-r\|)$ to $\eta(\|t'-r\|)$ will increase. Thus if nodes only transmit to their closest neighbors, interference from other transmissions (with similar transmit power) will appear to be small by comparison. This potentially allows many simultaneous transmissions throughout the network. On the other hand, under Model B no matter how close t and r become, interference from other transmissions can be on a similar level. Therefore even if nodes transmit only to their closest neighbors, interference from other transmissions may still be significant. This limits the number of simultaneous transmissions, which in turn leads to different results on the throughput scaling of a network. For example, as mentioned in the introduction, [8] showed that when the network region A remains fixed, the per-node throughput under Model B is $\Theta(\frac{1}{n})$, which is quite different than the $\Omega(\frac{1}{\sqrt{n \log n}})$ found in [1]. This is precisely because the interference prevents the number of simultaneous transmissions from growing to infinity due to the boundedness of the received power under Model B, so that nodes can only, in effect, use a time-division schedule. A similar result was found in [10] for all propagation models that are bounded at the origin.

On the other hand, if the underlying asymptotic regime is such that the node density is kept constant as n increases, i.e. if $\gamma = \frac{1}{2}$, then the difference between Model A and Model B discussed above will not effect the resulting scaling laws of the network (see Table 1).

3.2 Success Criterion

In this paper, we adopt the *SINR (signal to interference and noise ratio) criterion* [1], which is commonly used for this purpose. To introduce it, consider the situation that a node at t and all nodes at locations in $T = \{t_1, \dots, t_{M-1}\}$ transmit simultaneously in a given slot, that the transmission from t is intended to be received at r , that the received powers at r from the transmitters at t and T are P' and $\{P_1, \dots, P_{M-1}\}$, respectively, and that background noise with power N_o is also received. Let $\mathcal{T} = \mathbb{R}^2 \times \mathbb{R}^2 \times \overline{\mathbb{R}^2}$.

Definition 3 - SINR

The signal to interference noise ratio (SINR) at r is

$$\text{SINR}(P', \{P_1, \dots, P_{M-1}\}, N_o) = \frac{P'}{N_o + \sum_{i=1}^{M-1} P_i}.$$

When a propagation model η is available, then with a small abuse of notation, SINR becomes a function of $(t, r, T) \in \mathcal{T}$ and P , as well as N_o :

$$\text{SINR}(t, r, T, P, N_o, \eta) = \text{SINR}(P\eta(\|t-r\|), \{P\eta(\|t_1-r\|), \dots, P\eta(\|t_{M-1}-r\|)\}, N_o).$$

Note that with the above definition, transmissions at all locations in T are considered noise. We now use the above SINR functions to characterize a power indicator function

$$\phi(P', \{P_1, \dots, P_{M-1}\}, N_o) = \begin{cases} 1, & \text{SINR}(P', \{P_1, \dots, P_{M-1}\}, N_o) \geq \beta \\ 0, & \text{else} \end{cases},$$

and the success indicator function based on ϕ

$$\psi(t, r, T, P, N_o, \eta) = \begin{cases} 1, & \text{SINR}(t, r, T, P, N_o, \eta) \geq \beta \\ 0, & \text{else} \end{cases}.$$

The success criterion to be used in this paper is that determined by ψ above, as summarized below.

Definition 4 - SINR Success Criterion

Given $\beta > 0$, $N_o > 0$, $(t, r, T) \in \mathcal{T}$ and propagation model η (either Model A or B with associated parameter α), a transmission with power P from t to r in the presence of background noise with power N_o and interfering transmitters at the locations in T , each with power P , is said to be SINR_β -successful at power P (or we say (t, r, T) satisfies the SINR_β criterion at power P) if

$$\text{SINR}(t, r, T, P, N_o, \eta) \geq \beta,$$

This criterion is called the physical model in [1].

Note that in the next section, unless explicitly stated, we do not restrict the transmitters and receivers to lie in any specified region such as a disk.

4 Distance Based Success Criteria

To design a successful system \mathcal{S}_n , one must design a set of compatible scheduled routes such that all induced hop sets $\mathcal{H}_1, \dots, \mathcal{H}_p$ are successful with respect to the SINR_β criterion. While it is straightforward to check if any candidate hop set is successful, it is not at all clear how one goes about designing a hop set to be successful. To facilitate such design, Gupta and Kumar [1] introduced a concept that we refer to as a *distance-based* success criterion. This is a criterion that can be tested knowing only distances between transmitters, and between transmitters and receivers. Specifically, Gupta and Kumar first introduced a distance-based criterion called the *protocol model*, which is specified by two parameters ρ and Δ and which declares that a transmission from t to r is successful in the presence of other transmitters in T , i.e., $\psi(t, r, T, P, N_o, \eta) = 1$, if $\|t - r\| \leq \rho$ and $\|t' - r\| \geq \rho(1 + \Delta)$ for all $t' \in T$. However, in deriving constructive results, [1] used a distance-based criterion of the following form, which we find more useful.

Definition 5 - Distance-Based Success Criterion $\text{DC}(C, D)$

Given $C, D > 0$, the transmission from t to r in the presence of transmissions from the locations in the finite set T is said to be $\text{DC}(C, D)$ -successful (or we say $(t, r, T) \in \mathcal{T}$ satisfies the $\text{DC}(C, D)$ criterion) if

$$\|t - r\| \leq C$$

and

$$\|t' - t''\| \geq C(2 + D) \text{ for all } t', t'' \in T \cup \{t\}.$$

Notice that there is no dependence on power, only on internode distances. Notice also that instead of requiring $\|t' - r\|$ to be large for $t' \neq t$ (as in the protocol model), this criterion requires $\|t' - t''\|$ to be large. However, the triangle inequality implies that if t, r, T satisfies the $\text{DC}(C, D)$ criterion, then $\|r - t'\| > C(1 + D)$ for all $t' \neq t$. Moreover, the $\text{DC}(C, D)$ criterion has the effect of constraining the density of transmitters in T , for reasons to be explained shortly.

Since the SINR_β criterion is the preferred success criterion that we actually wish each system to satisfy, but a $\text{DC}(C, D)$ criterion is one that we can tractably design systems to satisfy, we will want to choose C and D so that the SINR_β criterion is satisfied whenever the $\text{DC}(C, D)$ criterion is satisfied. In this case, (C, D) are said to *ensure* the SINR_β criterion, as defined below.

Definition 6 Given N_o and η , a pair (C, D) is said to ensure the SINR_β criterion under propagation model η , if there exists $P > 0$ such that any $(t, r, T) \in \mathcal{T}$ that satisfies the $\text{DC}(C, D)$ criterion also satisfies the SINR_β criterion at power P , i.e $\psi(t, r, T, P, N_o) = 1$.

The reason that $\text{DC}(C, D)$ criterion constrains the distance between every pair of nodes in $T \cup \{t\}$, as opposed to the distance between each transmitter in T and r , is that it limits the density of transmitters in T , which in the context of the SINR_β criterion, limits the total interfering power at r , and enables a $\text{DC}(C, D)$ criterion to ensure SINR_β .

The following lemma, whose proof uses techniques similar to those used in [1], will be used later to find (C, D) that ensure SINR_β . From now on, unless otherwise stated, we assume Propagation Model B, $\eta(d) = \frac{1}{(1+d)^\alpha}$. A similar result could be obtained for Propagation model A.

Lemma 1 For given $N_o > 0$, $\alpha > 0$, and the corresponding η given by Propagation Model B, if $(t, r, T) \in \mathcal{T}$ satisfies the $\text{DC}(C, D)$ criterion, then for all $P > 0$,

$$\text{SINR}(t, r, T, P, N_o, \eta) \geq \frac{1}{(1+C)^\alpha \left(\frac{N_o}{P} + \sum_{k=1}^K \frac{6k+3}{(1+kC(1+D/2))^\alpha} \right)}, \quad (1)$$

where $K = \left\lfloor \frac{\max\{\|t'-r\|: t' \in T\}}{C(1+D/2)} \right\rfloor$.

Proof: We are given that $(t, r, T) \in \mathcal{T}$ satisfies $\text{DC}(C, D)$ and that a transmitter at t wishes to transmit to r with power P in the presence of noise power N_o and simultaneous transmitters, each with power P , at the locations in T . We prove the lemma by upper bounding the number of transmitters in T in circular rings centered at r , and using the propagation law to upper bound the power received from them. Accordingly, let $\delta = C(1 + D/2)$, and let T_k denote the subset of transmitters in T whose distance from r is larger than $k\delta$ and no larger than $(k+1)\delta$, $k \in \{1, 2, \dots, K\}$, with $K = \left\lfloor \frac{\max\{\|t'-r\|: t' \in T\}}{\delta} \right\rfloor$. From the $\text{DC}(C, D)$ criterion and the triangle inequality it follows easily that no transmitter in T lies within δ of r . Note also that K has been chosen large enough that every transmitter in T is included in one of the T_k 's.

The number of transmitters in T_k , denoted $|T_k|$, can now be bounded from above by the area of the circular ring, illustrated in Figure 1, with outer radius $(k+2)\delta$ and inner radius $(k-1)\delta$ divided by the area of a circle of radius δ . This is because the $\text{DC}(C, D)$ criterion implies that circles of radius δ centered about each transmitter in T_k do not overlap, and because the circles corresponding to transmitters in T_k lie within the aforementioned ring. It follows that

$$|T_k| \leq \frac{\pi(k+2)^2\delta^2 - \pi(k-1)^2\delta^2}{\pi\delta^2} = 6k+3.$$

According to the propagation law, the power received at r from a transmitter in T_k is at most $P/(1+k\delta)^\alpha$. And since $\text{DC}(C, D)$ implies the power received from t at r is at least $P/(1+C)^\alpha$, we have

$$\begin{aligned} \text{SINR}(t, r, T, P, N_o, \eta) &= \frac{P\eta(\|t-r\|)}{N_o + \sum_{k=1}^K \sum_{t' \in T_k} P\eta(\|t'-r\|)} \geq \frac{\frac{P}{(1+C)^\alpha}}{N_o + \sum_{k=1}^K |T_k| \frac{P}{(1+k\delta)^\alpha}} \\ &\geq \frac{1}{(1+C)^\alpha \left(\frac{N_o}{P} + \sum_{k=1}^K \frac{6k+3}{(1+k\delta)^\alpha} \right)}, \end{aligned}$$

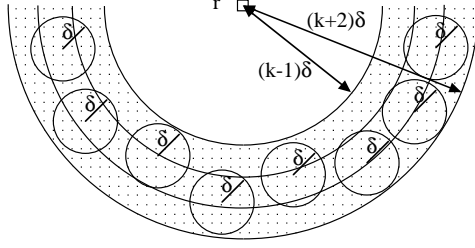


Figure 1: The shaded region is the ring surrounding r containing circles of radius δ centered on the transmitters in T_k .

which completes the proof of the lemma. \square

Notice that for any C and D , one can choose P so large that the term N_o/P in the denominator of (1) is negligible. We therefore obtain the following.

Corollary 2 (C, D) ensures the $SINR_\beta$ criterion under Propagation Model B if

$$(1 + C)^\alpha \sum_{k=1}^{\infty} \frac{6k + 3}{(1 + kC(1 + D/2))^\alpha} < \frac{1}{\beta}. \quad (2)$$

This corollary can be strengthened somewhat if it is known that all transmitters and receivers lie in some bounded region A . Specifically, one can easily extend the proof to show that for (C, D) to ensure the $SINR_\beta$ criterion, it suffices for (2) to hold with infinity as the upper limit of the sum replaced by $\left\lfloor \frac{\text{diam}(A)}{C(1+D/2)} \right\rfloor$.

The following lemma provides examples of (C, D) that ensure the $SINR_\beta$ criterion.

Lemma 3 Let $\beta > 0$, $\alpha > 2$, and consider the corresponding Propagation Model B.

(a) (C, D) ensures $SINR_\beta$ if

$$\frac{(1 + C)}{C(2 + D)} < \frac{1}{\tau \beta^{1/\alpha}} \quad (3)$$

where $\tau = 2 \left(\sum_{k=1}^{\infty} \frac{6}{k^{\alpha-1}} + \sum_{k=1}^{\infty} \frac{3}{k^\alpha} \right)^{1/\alpha}$.

(b) For any $C > 0$, there exists $D > 0$, depending on α , β and C such that (C, D) ensures $SINR_\beta$.

(c) There exists $D > 0$, depending on α and β , such that (C, D) ensures $SINR_\beta$ for all sufficiently large C .

Proof: Let $\alpha > 2$, $\beta > 0$. Dropping a “1” from the denominator in the left side of (2) yields the upper bound

$$(1 + C)^\alpha \sum_{k=1}^{\infty} \frac{6k + 3}{(kC(1 + D/2))^\alpha} = \frac{(1 + C)^\alpha}{(C(2 + D))^\alpha} 2^\alpha \left(\sum_{k=1}^{\infty} \frac{6}{k^{\alpha-1}} + \sum_{k=1}^{\infty} \frac{3}{k^\alpha} \right).$$

Since $\alpha > 2$, both series on the right hand side are finite. By Corollary 2 if the above is less than $\frac{1}{\beta}$, which is equivalent to (3), then (C, D) ensures $SINR_\beta$. This shows (a). Part (b) follows directly from (a). Part (c) follows from (a) and the fact that for all sufficiently large C , $\frac{1+C}{C} \leq 2$. \square

For large throughput, we would like C and D to be small in order to permit a large number of simultaneously transmitting nodes, as will be evident in the proof of Theorem 5. On the other hand, they must ensure the SINR_β criterion, and the next lemma demonstrates, not surprisingly, that when C and D are small, $\text{DC}(C, D)$ does not ensure the SINR_β criterion. Such limitations on C and D are what limit the attainable throughput. This lemma might also have future use in deriving upper bounds to attainable throughput. The proof is given in Appendix A.

Lemma 4 *Let N_o and Propagation Model B with parameter $\alpha > 2$ be given.*

(a) *Given $C, D > 0$ and a positive integer m , there exists $(t, r, T) \in \mathcal{T}$ such that T has m members, $\text{DC}(C, D)$ is satisfied, and for any $P > 0$,*

$$\text{SINR} \leq \frac{1}{7} \left(\frac{1 + 2C(2 + D)}{1 + C} \right)^\alpha \left(\sum_{k=1}^{\lfloor \sqrt{m/7} - 2 \rfloor} \frac{1}{k^{\alpha-1}} \right)^{-1}. \quad (4)$$

(b) *If*

$$\frac{1}{7} \left(\frac{1 + 2C(2 + D)}{1 + C} \right)^\alpha \left(\sum_{k=1}^{\infty} \frac{1}{k^{\alpha-1}} \right)^{-1} < \beta \quad (5)$$

then (C, D) does not ensure SINR_β , i.e., there exists $(t, r, T) \in \mathcal{T}$ such that $\text{DC}(C, D)$ is satisfied but for all $P > 0$, SINR_β is not.

(c) *If*

$$\frac{1}{7} \left(\sum_{k=1}^{\infty} \frac{1}{k^{\alpha-1}} \right)^{-1} < \beta \quad (6)$$

then all sufficiently small C, D do not ensure SINR_β .

In summary, the distance-based criterion $\text{DC}(C, D)$ can be used as an intermediary that facilitates the design of systems that satisfy an SINR_β criterion. To do so, one needs to choose C, D appropriately, for example, as indicated in Lemma 3. The choice depends significantly on the propagation model. For example, when n nodes are distributed over a region A with unit area, one may conclude from [1] that $C_n = \sqrt{\frac{\ln n}{\pi n}}$ and D_n equal to an appropriate constant ensures SINR_β under Propagation Model A, whereas [11] shows this does not hold under Propagation Model B.

5 The Principal Result

The following is the principal result of this paper.

Theorem 5 *Consider the many-to-many communication task for a set of n source-destination pairs \mathcal{P}_n randomly distributed over a disk of radius n^γ , $\gamma \geq 0$, with a propagation model of the form $\eta(d) = \frac{1}{(1+d)^\alpha}$ with $\alpha > 2$, an SINR_β success criterion with parameter $\beta > 0$, and background noise with power N_o . Then there exist constants $c_5, \bar{c}_5 > 0$ depending only on α, β such that for any n and any source-destination set \mathcal{P}_n , there exists a many-to-many system $\mathcal{S}_n(\mathcal{P}_n)$ and a power P_n with throughput $\lambda_n(\mathcal{P}_n)$ such that for any packet transmission rate $W > 0$ (bits per slot), as $n \rightarrow \infty$*

$$\Pr(\mathcal{S}_n(\mathcal{P}_n) \text{ is } \text{SINR}_\beta\text{-successful}) \rightarrow 1$$

and if $\gamma < \frac{1}{2}$,

$$\Pr \left(\lambda_n(\mathcal{P}_n) \geq \frac{W}{c_5 n^{1-\gamma}} \right) \rightarrow 1, \quad (7)$$

whereas if $\gamma \geq \frac{1}{2}$,

$$\Pr \left(\lambda_n(\mathcal{P}_n) \geq \frac{W}{c_5 \sqrt{n \ln n}} \right) \rightarrow 1. \quad (8)$$

When $\gamma < \frac{1}{2}$, P_n depends on α, β, N_o , but not n, \mathcal{P}_n or γ . When $\gamma \geq \frac{1}{2}$, P_n depends on α, β, N_o , increases with n, γ , and does not depend on \mathcal{P}_n .

To prove this theorem, we first prove a result like the above, but with the SINR success criterion replaced by a distance-based success criterion. The previous theorem will then be proven by appropriate choices of the parameters of the distance-based criterion.

Theorem 6 *Consider the many-to-many communication task for a set of n source-destination pairs \mathcal{P}_n randomly distributed over a disk of radius n^γ , $\gamma \geq 0$, with a distance-based success criterion. For each n , let C_n be chosen so that*

$$\frac{C_n}{n^\gamma} < \frac{1}{2} \text{ for all sufficiently large } n \quad (9)$$

and

$$a n \left(\frac{C_n}{n^\gamma} \right)^2 + \ln \frac{C_n}{n^\gamma} \rightarrow \infty \quad (10)$$

where $a = \frac{1}{2^{13}\pi} \ln \frac{e}{2}$. Then for any n , any $D_n > 0$ and any source-destination set \mathcal{P}_n , there exists a many-to-many system $\mathcal{S}_n(\mathcal{P}_n)$, with throughput $\lambda_n(\mathcal{P}_n)$, such that for any packet transmission rate $W > 0$ (bits per slot), as $n \rightarrow \infty$

$$\Pr (\mathcal{S}_n(\mathcal{P}_n) \text{ is } DC(C_n, D_n)\text{-successful}) \rightarrow 1 \quad (11)$$

and

$$\Pr \left(\lambda_n(\mathcal{P}_n) \geq \frac{W}{c_6 n^{1-\gamma} C_n (2 + D_n)^2} \right) \rightarrow 1. \quad (12)$$

where $c_6 = 27 \times 2^{14}$.

Note that we have not attempted to minimize the constant c_6 . Note also that there is no restriction on D_n . Reducing D_n permits (12) to guarantee a higher throughput, up to a point of diminishing returns. However, when we apply this result to prove Theorem 5, we will see, not surprisingly, that if D_n is too small, the distance-based success criterion will not ensure the SINR_β criterion.

We now comment on conditions (9) and (10). The first places a natural upper bound on C_n as half the radius of the network region A_n . For example, when $\gamma = 0$, it requires $C_n \leq \frac{1}{2}$. The second prevents C_n from being too small. We note that the expression in (10) is a monotonically increasing function of C_n . It is satisfied, for example, if $\frac{C_n}{n^\gamma} = b \left(\frac{\ln n}{n} \right)^{1/2}$ and $b \geq \frac{1}{\sqrt{2a}}$, but not if $b < \frac{1}{\sqrt{2a}}$. When applying Theorem 6 to prove Theorem 5, it turns out that (10) comes into play only when $\gamma \geq \frac{1}{2}$. In addition, the proof of Theorem 6 will demonstrate that conditions (9) and (10) are sufficient to ensure that as n increases, with probability approaching one, the wireless

network formed by the nodes is *connected*, in the sense that there is a route from every node to every other node with all hops having length C_n or less. Gupta and Kumar [12] found a necessary and sufficient condition for such asymptotic connectivity that, when the network region A_n has radius n^γ , reduces to $n(\frac{C_n}{n^\gamma})^2 - \ln n \rightarrow \infty$. Since (9) and (10) are sufficient to ensure asymptotic connectivity, they must of course imply the complete connectivity condition of [12], as well. We make a direct verification of this in Appendix B. We also note that $\frac{C_n}{n^\gamma} = b(\frac{\ln n}{n})^{1/2}$ satisfies the connectivity condition for every $b > 1$, whereas (10) is satisfied only if b is at least $\frac{1}{\sqrt{2a}}$, which is much larger than one. However, this difference is not significant in determining the rate of throughput scaling.

In the remainder of this section we prove Theorem 6; then use the latter to prove Theorem 5. We also describe how it can be used to derive the $\Omega(\frac{1}{\sqrt{n \ln n}})$ scaling result of [1].

Proof of Theorem 6: We follow an approach similar in many respects to that of [1]. Given $\gamma \geq 0, W, n$, a source-destination set \mathcal{P}_n , and C_n, D_n satisfying (9) and (10), we need to construct a system that with high probability is successful and has the desired throughput. The proof is divided into steps. The first three describe a procedure for designing a system for a particular n and \mathcal{P}_n ; the remaining steps derive the performance of the designed system. Specifically, Step 1 chooses a route for each source-destination pair, i.e. a sequence of hops from the source to the destination. This is done in a way that will make it possible to show that all hops have length less than or equal to C_n , with probability approaching one as n tends to infinity, where probability is with respect to the random source-destination set. Moreover, these routes are chosen in a way that attempts to limit L_n , the maximum number of routes assigned to any one node. Step 2 chooses a collection of potential transmitter sets T_1, \dots, T_{S_n} such that the members of each set are at least $C_n(2 + D_n)$ apart from one another (so that according to the $\text{DC}(C_n, D_n)$ criterion, they may simultaneously and successfully transmit to receivers at distance C_n or less), and every node is included in exactly one set (so that every node is permitted to transmit). These sets are chosen with the goal of minimizing S_n . Step 3 combines the routes of Step 1 and the potential transmitter sets of Step 2 to form compatible scheduled routes, i.e. a system, with period $p_n = L_n S_n$ and throughput $\lambda_n = \frac{W}{L_n S_n}$. This system will be $\text{DC}(C_n, D_n)$ -successful if and only if all hops have length C_n or less, as was the goal of Step 1. Step 4 shows this to be the case with probability approaching one as $n \rightarrow \infty$. In addition, it shows that $L_n = \mathcal{O}(\frac{n^\gamma}{C_n})$ with probability approaching one. Step 5 shows that $S_n = \mathcal{O}(n^{1-2\gamma} C_n^2 (2 + D_n)^2)$ with probability approaching one as $n \rightarrow \infty$. Step 6 completes the proof by using the results of Steps 4 and 5 to show (11) and (12).

Step 1: Route selection

Given n and C_n , let $z = \frac{C_n}{2}$, and partition the network region A_n , which is a disk of radius n^γ , into convex cells, each having diameter at most z and area at least μz^2 , where $\mu > 0$ is some constant that does not depend on n, γ , or z . While it is intuitively clear that this can be done, Lemma C1 of Appendix C provides a concrete proof with $\mu = \frac{1}{512}$. It requires the radius n^γ to be at least $\frac{z}{4}$, which holds because of the choice $z = \frac{C_n}{2}$ and the assumption (9) that $C_n < \frac{n^\gamma}{2}$. The number of cells in the partition, denoted M_n , is at most $\frac{\pi n^{2\gamma}}{\mu z^2}$.

For each (s, d) in the source-destination set \mathcal{P}_n , draw a straight line from s to d . Form a route for this pair by following the line from s to d and selecting one node, and a corresponding hop, from each cell intersected by the line, whenever there is such a node. Convexity of the cells ensures that

the line does not pass through the same cell twice. The fact that cells have diameters no larger than z implies that if there is at least one node in each cell intersected by the line, then the length of each hop is at most $2z = C_n$. Ordinarily, there will be more than one node in a given cell, a fact that can be used to reduce the likelihood that a node is assigned to too many routes. Indeed, if X_i source-destination lines intersect the i th cell and this cell contains Y_i nodes, we apportion the X_i routes as equally as possible among the Y_i nodes. Thus, each node is assigned to no more than $\left\lceil \frac{X_i}{Y_i} \right\rceil$ routes. If there are no nodes in i th cell, then we take $\left\lceil \frac{X_i}{Y_i} \right\rceil$ to be infinity, even if $X_i = 0$. (We will show later that $\Pr(\min_i Y_i = 0) \rightarrow 0$ as $n \rightarrow \infty$.) Let L_n denote the maximum of $\left\lceil \frac{X_i}{Y_i} \right\rceil$ over all cells.

Step 2: Potential transmitter sets

We begin by forming a graph with the n nodes as the vertices and an edge between any pair of nodes separated by $C_n(2 + D_n)$ or less. Let S_n denote the maximum number of edges connected to any one node. We use the graph coloring theorem [13, 14] to assign one of S_n distinct colors to each node in such a way that no two nodes connected by an edge receive the same color⁵. We then partition the n nodes into S_n transmitter sets T_1, \dots, T_{S_n} according to their assigned colors. Since each node in one transmitter set is separated by $C_n(2 + D_n)$ from every other node in the set, simultaneous transmissions by all nodes in the set to receivers located within C_n of each transmitter will be DC(C_n, D_n)-successful.

Step 3: System

We now form a system $\mathcal{S}_n(\mathcal{P}_n)$ with period $p(n) = L_n S_n$. Recall the routes found in Step 1 and the groups of potential transmitters, T_1, \dots, T_{S_n} , found in Step 2. Consider the sequence of $L_n S_n$ transmitter sets $\bar{T}_1, \dots, \bar{T}_{L_n S_n} = T_1, \dots, T_{S_n}, T_1, \dots, T_{S_n}, \dots, T_1, \dots, T_{S_n}$. We now schedule the hops of the routes in “rounds”. In the first round, for $j = 1, \dots, S_n$, and for each node in \bar{T}_j , select a route to which it is assigned (if any) and schedule the corresponding hop from this node for time slot j . In the second round, for $j = S_n + 1, \dots, 2S_n$, and for each node in \bar{T}_j , select a route (if any) to which it is assigned that was not selected in the previous round, and schedule the corresponding hop from this node for time slot j . We continue in this way for $L_n - 2$ further rounds. Since each node of each route is assigned to at most L_n routes, each of its assigned hops will be assigned a time slot. The resulting scheduled routes are compatible, because in any time slot each node is assigned to only one route. In summary, we have created a system $\mathcal{S}_n(\mathcal{P}_n)$ with period $p(n) = L_n S_n$ and throughput

$$\lambda_n(\mathcal{P}_n) = \frac{W}{L_n S_n}.$$

Note that the use of a partition and straight lines to define the routes in Step 1 is just as in [1], except that we describe how to partition a disk, rather than the surface of a sphere. Unlike [1], we apportion the load as equally as possible among the nodes in each cell. Because of this, in Step 2 we needed to color a graph with one vertex for each node, in contrast to [1], which colored a graph with one vertex for each cell. Since our graph has more nodes, it requires more colors, i.e., a larger S_n . However, the analysis in Step 4 is simplified, because it is easier to determine which nodes interfere with one another than which cells interfere with one another. Moreover, the throughput

⁵Actually, $S_n - 1$ colors are sufficient, but we use S_n to simplify expressions.

is not affected because each node in our system is responsible for correspondingly fewer routes, i.e., a smaller L_n , than each cell in the system design of [1].

Step 4: $L_n = \mathcal{O}(\frac{n^\gamma}{C_n})$ with high probability

Recall that X_i denotes the number of source-destination lines that intersect the i th cell of the partition chosen in Step 1, and Y_i denotes the number of nodes in the i th cell. The following lemma provides a bound to $L_n = \max_i \left\lceil \frac{X_i}{Y_i} \right\rceil$ that applies with probability approaching one. Notice that when a source-destination line intersects a cell, say i , that has no nodes, then $Y_i = 0$, and by the convention of Step 1, $L_n = \infty$. Therefore, the result of the lemma below also implies that with probability approaching one, all hops of all routes are no longer than $2z = C_n$. This, in turn, implies the connectivity mentioned in the discussion after the theorem statement.

Lemma 7 *Under the conditions of Theorem 6,*

$$\Pr \left(L_n \leq c_7 \frac{n^\gamma}{C_n} + 1 \right) \rightarrow 1, \quad \text{as } n \rightarrow \infty$$

where $c_7 = 3 \times 2^{13} \pi$.

Proof: Since $L_n \triangleq \max_i \left\lceil \frac{X_i}{Y_i} \right\rceil \leq \frac{\max_i X_i}{\min_i Y_i} + 1$, we will separately consider the behavior of $\max_i X_i$ and $\min_i Y_i$, finding quantities $a(n), b(n) > 0$ such that $\max_i X_i$ exceeds $a(n)$ with vanishing probability, $\min_i Y_i$ is less than $b(n)$ with vanishing probability, and $\frac{a(n)}{b(n)} \leq c_7 \frac{n^\gamma}{C_n}$.

We begin by writing $Y_i = \sum_{j=1}^n B_{i,j}$, where $B_{i,j} = 1$ when the j th node lies in the i th cell of the partition, and $B_{i,j} = 0$ otherwise. By the model for the random location of nodes, for each i , $B_{i,1}, \dots, B_{i,n}$ are IID with

$$q_{n,i} \triangleq \Pr(B_{i,j} = 1) = E[B_{i,j}] = \frac{\text{area of } i\text{th cell}}{\text{area of network region}} \geq \frac{\mu C_n^2}{4\pi n^{2\gamma}} \triangleq q_n.$$

It follows that $E[Y_i] \geq nq_n = \frac{\mu C_n^2}{4\pi} n^{1-2\gamma}$. Applying the union bound yields

$$\Pr \left(\min_i Y_i < \frac{1}{2} nq_n \right) \leq \sum_{i=1}^{M_n} \Pr \left(Y_i < \frac{1}{2} nq_n \right) \quad (13)$$

Similarly, write $X_i = \sum_{j=1}^n A_{i,j}$, where $A_{i,j} = 1$ when the line from s_j to d_j passes through the i th cell, and $A_{i,j} = 0$ otherwise. By the model for the random location of sources and random choices of destinations, for each i , $A_{i,1}, \dots, A_{i,n}$ are independent and identically distributed (IID), with $p_{n,i} \triangleq \Pr(A_{i,j} = 1) = E[A_{i,j}]$. Note that the X_i 's are not identically distributed. For example, $p_{n,i}$ is larger for a cell near the center of the disk than one near the edge. Nevertheless, Lemma C2 of Appendix C finds a common upper bound to all $p_{n,i}$'s, namely,

$$p_{n,i} \leq 3 \frac{C_n}{n^\gamma} \triangleq p_n$$

for all i . (Lemma C2 requires $z \leq n^\gamma$, i.e. $C_n \leq 2n^\gamma$, which is guaranteed by (9).) It follows that $E[X_i] \leq np_n = 3C_n n^{1-\gamma}$. Once again we apply the union bound.

$$\Pr \left(\max_i X_i > 2np_n \right) \leq \sum_{i=1}^{M_n} \Pr (X_i > 2np_n) \quad (14)$$

The facts that $E[Y_i] \geq nq_n$ and $E[X_i] \leq np_n$ and that $\min_i Y_i$ and $\max_i X_i$ have mean values in the vicinity of nq_n and np_n , respectively, suggests that the probabilities appearing in the summations in (13) and (14) are tail probabilities. Accordingly, they can be effectively bounded above using Chernoff bound techniques. From, Lemma C3 of Appendix C, which uses the Chernoff bound, we have

$$\Pr\left(Y_i < \frac{1}{2}nq_n\right) \leq \exp\left\{-nq_n\left(\frac{1}{2}\ln\frac{1}{2e} + 1\right)\right\} = \exp\left\{-\frac{1}{2}nq_n\ln\frac{e}{2}\right\}$$

and

$$\Pr(X_i > 2np_n) \leq \exp\left\{-np_n\left(2\ln\frac{2}{e} + 1\right)\right\} = \exp\left\{-np_n\ln\frac{4}{e}\right\} \quad (15)$$

(Application of Lemma C3 requires $p_n < 1$ and $q_n < 1$, both of which are implied by (9).)

Substituting, the above into (13) and (14), respectively, gives

$$\begin{aligned} \Pr\left(\min_i Y_i < \frac{1}{2}nq_n\right) &\leq \sum_{i=1}^{M_n} \exp\left\{-\frac{1}{2}nq_n\ln\frac{e}{2}\right\} = \exp\left\{-\frac{1}{2}nq_n\ln\frac{e}{2} + \ln M_n\right\} \\ &\leq \exp\left\{-\frac{1}{2}n\frac{\mu C_n^2}{4\pi n^{2\gamma}}\ln\frac{e}{2} + \ln\frac{4\pi n^{2\gamma}}{\mu C_n^2}\right\} \\ &= \exp\left\{-2an\left(\frac{C}{n^\gamma}\right)^2 - 2\ln\frac{C_n}{n^\gamma} + \ln\frac{4\pi}{\mu}\right\} \\ &\rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned} \quad (16)$$

where $a = \frac{1}{2^{13}\pi}\ln\frac{e}{2}$, and

$$\begin{aligned} \Pr\left(\max_i X_i > 2np_n\right) &\leq \sum_{i=1}^{M_n} \exp\left\{-np_n\ln\frac{4}{e}\right\} = \exp\left\{-np_n\ln\frac{4}{e} + \ln M_n\right\} \\ &\leq \exp\left\{-n\frac{C_n}{n^\gamma}3\ln\frac{4}{e} + \ln\frac{4\pi n^{2\gamma}}{\mu C_n^2}\right\} \\ &= \exp\left\{-n\frac{C_n}{n^\gamma}3\ln\frac{4}{e} - 2\ln\frac{C_n}{n^\gamma} + \ln\frac{4\pi}{\mu}\right\} \\ &\rightarrow 0, \text{ as } n \rightarrow \infty \end{aligned} \quad (17)$$

where the convergence to zero in (16) follows from condition (10), and the convergence to zero in (17) follows from (10) and the facts that $\frac{C_n}{n^\gamma} \geq \left(\frac{C_n}{n^\gamma}\right)^2$ (because (9) implies $\frac{C_n}{n^\gamma} < 1$) and that $3\ln\frac{4}{e} > 2a$.

We now combine results. With $c_7 = 3 \times 2^{13}\pi$,

$$\begin{aligned} \Pr\left(L_n \leq c_7\frac{n^\gamma}{C_n} + 1\right) &= \Pr\left(\max_i \left[\frac{X_i}{Y_i}\right] \leq \frac{2np_n}{\frac{1}{2}nq_n} + 1\right) \\ &\geq \Pr\left(\max_i \frac{X_i}{Y_i} \leq \frac{2np_n}{\frac{1}{2}nq_n}\right) \\ &\geq \Pr\left(\max_i X_i \leq 2np_n, \min_i Y_i \geq \frac{1}{2}nq_n\right) \\ &\rightarrow 1 \text{ as } n \rightarrow \infty, \end{aligned} \quad (18)$$

where the convergence to one follows from (16) and (17). This completes the proof of Lemma 7. \square

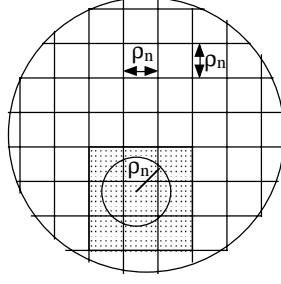


Figure 2: A circle of radius ρ_n is contained in a shaded $3\rho_n \times 3\rho_n$ square.

Step 5: $S_n = \mathcal{O}\left(n^{1-2\gamma}C_n^2(2+D_n)^2\right)$ with high probability

Recall that S_n equals the largest number of edges to which any node is connected. By the definition of the graph, S_n also equals the maximum, over all nodes, of the number of other nodes within $C_n(2+D_n)$ of the given node.

Lemma 8 *Under the conditions of Theorem 6,*

$$\Pr\left(S_n \leq \frac{18}{\pi}n\left(\frac{C_n(2+D_n)}{n^\gamma}\right)^2\right) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Proof: Let $\rho_n \triangleq C_n(2+D_n)$. As illustrated in Figure 2, overlay a square grid with sides of length ρ_n on the disk of radius n^γ . This partitions the network region into cells with area at most ρ_n^2 . The number of such cells, denoted \overline{M}_n , is at most $\frac{\pi(n^\gamma + \sqrt{2}\rho_n)^2}{\rho_n^2}$, because the \overline{M}_n squares of the grid that are contained in or intersect the disk of radius n^γ are all contained in a disk of radius $n^\gamma + \sqrt{2}\rho_n$. For $i = 1, \dots, \overline{M}_n$, let U_i denote the number of nodes that lie in a $3\rho_n \times 3\rho_n$ square centered on the i th cell of the partition. Since every circle of radius ρ_n lies in at least one of these $3\rho_n \times 3\rho_n$ squares, it follows that $S_n \leq \max_i U_i$. Let us also observe that $U_i = \sum_{j=1}^n B_{i,j}$, where $B_{i,j} = 1$ when the j th node lies in the $3\rho_n \times 3\rho_n$ square centered on the i th cell of the partition, and $B_{i,j} = 0$ otherwise. By the model for the random location of nodes, for each i , the $B_{i,j}$'s are IID with

$$p_{n,i} \triangleq \Pr(B_{i,j} = 1) = E[B_{i,j}] \leq \frac{\text{area of 9 squares centered on the } i\text{th cell}}{\text{area of network region}} = \frac{9\rho_n^2}{\pi n^{2\gamma}} \triangleq p_n.$$

(Note that $B_{i,j}$, $p_{n,i}$ and p_n are not the same as in the proof of the previous lemma.)

Proceeding as in (14), (15) and (17), we find

$$\begin{aligned} \Pr\left(S_n > \frac{18}{\pi}n^{1-2\gamma}\rho_n^2\right) &< \Pr\left(\max_i U_i > 2np_n\right) \leq \sum_{i=1}^{\overline{M}_n} \Pr(U_i > 2np_n) \\ &\leq \sum_{i=1}^{\overline{M}_n} \exp\left\{-np_n \ln \frac{4}{e}\right\} = \exp\left\{-np_n \ln \frac{4}{e} + \ln \overline{M}_n\right\} \\ &\leq \exp\left\{-n \frac{9\rho_n^2}{\pi n^{2\gamma}} \ln \frac{4}{e} + \ln \frac{\pi(n^\gamma + \sqrt{2}\rho_n)^2}{\rho_n^2}\right\} \\ &= \exp\left\{-2\left(n\left(\frac{\rho_n}{n^\gamma}\right)^2 \frac{9}{2\pi} \ln \frac{4}{e} - \ln\left(\frac{n^\gamma}{\rho_n} + \sqrt{2}\right)\right) + \ln \pi\right\} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned} \tag{19}$$

where the third inequality follows from Lemma C3 and the fact that $p_n \geq p_{n,i}$, and where the convergence is explained as follows. The principal terms in the last exponential above have the form $\bar{a}nv_n^2 - \ln\left(\frac{1}{v_n} + \sqrt{2}\right)$, where $\bar{a} = \frac{9}{2\pi} \ln \frac{4}{e}$ and $v_n = \frac{\rho_n}{n^\gamma} = \frac{C_n(2+D_n)}{n^\gamma} = u_n(2+D_n)$, and where $u_n = \frac{C_n}{n^\gamma}$. We now have

$$\begin{aligned} \bar{a}nv_n^2 - \ln\left(\frac{1}{v_n} + \sqrt{2}\right) &\geq \begin{cases} \bar{a}nv_n^2 - \ln \frac{2}{v_n}, & \frac{1}{v_n} \geq \sqrt{2} \\ \bar{a}nv_n^2 - \ln 2\sqrt{2}, & \text{else} \end{cases} \\ &= \begin{cases} \bar{a}nv_n^2 + \ln v_n - \ln 2, & \frac{1}{v_n} \geq \sqrt{2} \\ \bar{a}nv_n^2 - \ln 2\sqrt{2}, & \text{else} \end{cases} \end{aligned}$$

Since (10) shows $anu_n^2 + \ln u_n \rightarrow \infty$, and since $v_n > u_n$, $\bar{a} > a$, it follows that $\bar{a}nv_n^2 + \ln v_n \rightarrow \infty$. Moreover, (10) implies $nu_n^2 \rightarrow \infty$; so $nv_n^2 \rightarrow \infty$, as well. Using these facts in the above, shows that $\bar{a}nv_n^2 - \ln\left(\frac{1}{v_n} + \sqrt{2}\right) \rightarrow \infty$, which establishes the convergence to 0 in (19), and completes the proof of the lemma. \square

Note that in this step and the previous, we used the Chernoff bound to directly prove what was needed, rather than using the uniform convergence of the weak law of large numbers, as in [1].

Step 6: Completion of proof of Theorem 6

The system $\mathcal{S}_n(\mathcal{P}_n)$ has been designed so that it will be successful provided only that all hops have length C_n or less, which, as explained just before Lemma 7, happens if $L_n \leq c_7 \frac{n^\gamma}{C_n}$. Therefore, from Step 4,

$$\Pr(\mathcal{S}_n(\mathcal{P}_n) \text{ is DC}(C_n, D_n)\text{-successful}) \geq \Pr\left(L_n \leq c_7 \frac{n^\gamma}{C_n}\right) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Since $\lambda_n(\mathcal{P}_n) = \frac{W}{L_n S_n}$, from Steps 4 and 5, and the fact that $c_6 = c_7 \frac{18}{\pi}$, we have

$$\begin{aligned} \Pr\left(\lambda_n(\mathcal{P}_n) \geq \frac{W}{c_6 n^{1-\gamma} C_n (2+D_n)^2}\right) &= \Pr(L_n S_n \leq c_6 n^{1-\gamma} C_n (2+D_n)^2) \\ &\geq \Pr\left(L_n \leq c_7 \frac{n^\gamma}{C_n} \text{ and } S_n \leq \frac{18}{\pi} n \left(\frac{C_n(2+D_n)}{n^\gamma}\right)^2\right) \\ &\rightarrow 1, \text{ as } n \rightarrow \infty. \end{aligned}$$

This completes the proof of Theorem 6. \square

Proof of Theorem 5

Let $\alpha > 2$, $\beta > 0$ and $N_o > 0$ be given. To prove this theorem, for each n and for γ in two different ranges, we will make choices of C_n, D_n so that the following hold: (a) C_n satisfies (9) and (10), (b) (C_n, D_n) ensures SINR_β for all sufficiently large n , and (c) $c_6 n^{1-\gamma} C_n (2+D_n)^2$ reduces (in the limit) to the expression in the denominator of (7) or (8), as appropriate for the value of γ . Then for each n and \mathcal{P}_n , Theorem 6 will imply the existence of a system $\mathcal{S}_n(\mathcal{P}_n)$ that is $\text{DC}(C_n, D_n)$ successful with probability approaching one, whose throughput $\lambda_n(\mathcal{P}_n)$ satisfies (12). The fact that (C_n, D_n) ensures SINR_β for all sufficiently large n will imply the existence of a power P_n , depending on $\alpha, \beta, N_o, C_n, D_n, \gamma$, but not \mathcal{P}_n , such that the SINR_β criterion is satisfied, whenever the $\text{DC}(C_n, D_n)$ criterion is satisfied. Therefore, the fact that $\mathcal{S}_n(\mathcal{P}_n)$ is $\text{DC}(C_n, D_n)$ successful

with probability approaching one will imply that $\mathcal{S}_n(\mathcal{P}_n)$ is SINR_β successful with probability approaching one. Finally, (12) will imply (7) or (8) as appropriate.

It remains to make choices for C_n, D_n . When $0 \leq \gamma < \frac{1}{2}$, we choose $C_n = \frac{1}{4}$, which satisfies (9) and (10), and $D_n = \bar{C}_2$, where \bar{C}_2 is a value, depending only on α and β , such that $\text{DC}(\frac{1}{4}, \bar{C}_2)$ ensures SINR_β , whose existence is established by Lemma 3 Part (b). With $c_5 = c_6 \frac{1}{4} (2 + \bar{C}_2)^2$, it follows immediately that $c_6 n^{1-\gamma} C_n (2 + D_n)^2 = c_5 n^{1-\gamma}$, so that (7) holds. Since C_n, D_n are chosen to be constants, the power P_n can be chosen to be the same for all n and all $\gamma \in [0, \frac{1}{2})$. Thus the proof of the theorem is complete for $\gamma < \frac{1}{2}$.

When $\gamma \geq \frac{1}{2}$, we choose $C_n = n^{\gamma-\frac{1}{2}} \sqrt{\frac{2}{a} \ln n}$, $a = \frac{1}{2^{13}\pi} \ln \frac{e}{2}$, which satisfies (9) and (10). We also choose $D_n = \tilde{D}$, where \tilde{D} is a value, depending only on α and β , such that $\text{DC}(C, \tilde{D})$ ensures SINR_β for all sufficiently large C , whose existence is established by Lemma 3 Part (c). With $\bar{c}_5 = c_6 \sqrt{\frac{2}{a}} (2 + \tilde{D})^2$, it follows immediately that $c_6 n^{1-\gamma} C_n (2 + D_n)^2 = \bar{c}_5 \sqrt{n \ln n}$, so that (8) holds. When $\gamma \geq \frac{1}{2}$, $C_n \rightarrow \infty$, and it follows that P_n must also tend to infinity as n increases. Since C_n increases with γ , so too will P_n . This completes the proof of the theorem for $\gamma \geq \frac{1}{2}$. \square

We now comment on and justify the choices of C_n, D_n in the proof of Theorem 5. Clearly, to maximize throughput we want to choose them to minimize $C_n(2 + D_n)^2$ while satisfying (9), (10), and the requirement that (C_n, D_n) ensure SINR_β . Since we do not have a precise characterization of the (C_n, D_n) pairs that ensure SINR_β , we simply try to make the most of the sufficient conditions in Lemma 3. Note that since we seek only to maximize the “order” of the throughput, we need not attempt a precise minimization. To keep things simple, consider sequences C_n that either decrease to zero, tend to a constant, or increase to infinity, and consider the same three possibilities for D_n .

First, we cannot allow $C_n(2 + D_n)$ to go to zero because Lemma 4 shows that in this case for large n , (C_n, D_n) will not ensure SINR_β . Second, there is no point to making D_n go to zero, because the factor $(2 + D_n)$ cannot decrease below 2. Third, in the case of $\gamma < \frac{1}{2}$, there is no point to having one of C_n, D_n tend to infinity while the other remains finite, because we can satisfy the constraints with both taking finite values. It follows that for this case, the only potential competitor to the constant C_n, D_n that we chose in the proof of Theorem 5 is $C_n \rightarrow 0$ and $D_n \rightarrow \infty$. However, if $C_n \rightarrow 0$, then according to the sufficient condition of Lemma 3 Part (a), we need $D_n = \Omega(\frac{1}{C_n})$, so that $C_n(2 + D_n)^2 = C_n \Omega(\frac{1}{C_n^2}) \rightarrow \infty$, which of course is much worse than when C_n, D_n are chosen to be the constants in the proof of Theorem 5.

Next in the case of $\gamma \geq \frac{1}{2}$, in addition to the first two points above, we note that one cannot satisfy (10) unless $C_n \rightarrow \infty$. In the proof of Theorem 5 we chose C_n as small as possible and used Lemma 3 Part (c) to justify a constant choice of D_n . Making $D_n \rightarrow \infty$ would only reduce throughput. Therefore, for both ranges of γ , the choices of C_n, D_n in the proof are as good as we can make them with the available sufficient conditions for ensuring SINR_β .

In summary, we note that among the constraints on C_n, D_n , when $\gamma < \frac{1}{2}$, it is the requirement for ensuring SINR_β that limits throughput, whereas when $\gamma \geq \frac{1}{2}$, it is the connectivity-ensuring requirement (10) that limits throughput.

We conclude this section by noting that one can also use Theorem 6 to show straightforwardly that throughput $\Omega(\frac{1}{\sqrt{n \ln n}})$ is attainable for propagation model $\frac{1}{d^\alpha}$ and $\gamma \geq 0$. This demonstrates the original result of [1], as well as the fact that it applies for all $\gamma \geq 0$. To do so, one lets $C_n = n^{\gamma-\frac{1}{2}} \sqrt{\frac{2}{a} \ln n}$, $a = \frac{1}{2^{13}\pi} \ln \frac{e}{2}$, which satisfies (9) and (10), and one shows there is a \tilde{C}_2 such

that $\text{DC}(C_n, \tilde{C}_2)$ ensures the SINR_β criterion for the $\frac{1}{d^\alpha}$ propagation model. Substituting C_n and $D_n = \tilde{C}_2$ into (12) of Theorem 6 yields throughput $\Omega(\frac{1}{\sqrt{n \ln n}})$.

6 Concluding Remarks

In this paper we developed a constructive lower bound on attainable per-node throughput in a wireless network whose nodes are randomly distributed over a disk, with radius growing as n^γ with number of nodes n , for some $\gamma \geq 0$. By selecting $\gamma \in [0, \frac{1}{2})$, we can describe networks ranging from fixed size to fixed density. The lower bound has the form $\Omega(\frac{1}{n^{1-\gamma}})$ when $\gamma < \frac{1}{2}$, and $\Omega(\frac{1}{\sqrt{n \ln n}})$ when $\gamma \geq \frac{1}{2}$.

We now compare and contrast the approach used to derive our results to those used by Gupta and Kumar in [1]. First, recall that to prove Theorem 5, we first proved Theorem 6 for a distance-based success criterion and then chose the constants C_n and D_n to permit Theorem 6 to guarantee the largest possible throughput, while ensuring the SINR criterion. This is the strategy used in [1], except that it did not separate the derivation into two theorems, nor did it separate the discussion of how a distance-based criterion (which they called a protocol model) can ensure an SINR criterion, as we did in Section 4. We view that separating into two theorems and separating the discussion of distance-based criteria clarifies the derivation. We also indicated at the end of the previous section how the original $\Omega(\frac{1}{\sqrt{n \ln n}})$ result of [1] could be derived with our methods.

In [1], it was also shown that throughput of order $\Theta(\frac{1}{\sqrt{n \ln n}})$ is the best attainable when $\gamma = 0$, $\eta(d) = \frac{1}{d^\alpha}$, successful transmission occurs at rate W or 0 depending on whether received SINR is above or below a threshold β , and the system is *protocol based*, which in our terms means, essentially, that the system is designed so there are constants C and D that ensure the SINR_β criterion such that all hops of all routes have length C or less, and all simultaneous transmitters are at least $C(2 + D)$ apart from each other. Because of this and because our system is protocol based and designed in a similar fashion, it seems likely that the throughputs demonstrated by Theorem 5 are also order optimal among protocol-based systems. Indeed, for $\gamma < \frac{1}{2}$ they might be optimal among all systems, because the system that attains this throughput is not limited by the connectivity condition (10) of Theorem 6, whereas the throughput $\Omega(\frac{1}{\sqrt{n \ln n}})$ of [1] is clearly limited by the analogous connectivity constraint.

Appendix A

Proof of Lemma 4: (a) Suppose we are given β , N_o , Propagation Model of B with parameter $\alpha > 2$, $C, D > 0$ and a positive integer m . Let $\delta = C(2 + D)$. Consider the following specific choice of (t, r, T) . Let r be at the origin, let t be at distance C from the origin, and as illustrated in Figure 3, for $k = 1, 2, 3, \dots$, place as many nodes as possible on the circumference of a circle of radius $2k\delta$, subject to the constraint that nodes are δ apart, except that the Euclidean distance between the first and last chosen on the circle can be in $[\delta, 2\delta)$. Stop after placing a total of m nodes into T . Let T_k denote the nodes on the circle with radius $2k\delta$, let $|T_k|$ denote the number of nodes in it. Let K denote the number of rings into which we have placed nodes. Notice that any two nodes, whether on the same circle or not, are at least δ apart, except for t and r , which are C apart. Therefore, (t, r, T) satisfies $\text{DC}(C, D)$.

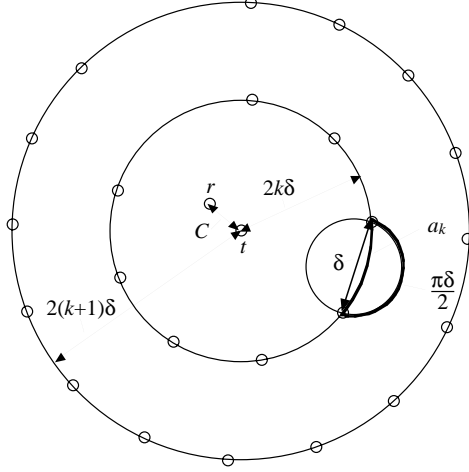


Figure 3: Illustration of t , r and the nodes in T_k and T_{k+1} for the proof of Lemma 4.

We now find bounds on $|T_k|$ and K . Let a_k denote the length of an arc on the circumference of a circle of radius $2k\delta$ between two points that are δ apart. Then

$$\delta < a_k \leq \frac{1}{2}\pi\delta$$

where the second inequality recognizes that the arc length is bounded above by half the circumference of a circle with diameter δ . Since $|T_k|a_k \leq 2\pi 2k\delta$, it follows from the above that

$$|T_k| \leq \frac{4\pi k\delta}{a_k} < 4\pi k < 14k .$$

Since for $k \leq K-1$, $(|T_k| + 1)a_k > 2\pi 2k\delta$, it follows that

$$|T_k| > \frac{4\pi k\delta}{a_k} - 1 \geq 8\pi k - 1 \geq 7k . \quad (\text{A1})$$

To bound K , we observe that

$$m = \sum_{k=1}^K |T_k| < \sum_{k=1}^K 14k = 14 \frac{K(K+1)}{2} < 7(K+1)^2 .$$

Therefore,

$$K > \sqrt{\frac{m}{7}} - 1 . \quad (\text{A2})$$

We now bound SINR. For any $P > 0$,

$$\begin{aligned}
\text{SINR}(t, r, T, P, N_o, \eta) &= \frac{\frac{P}{(1+\|t-r\|)^\alpha}}{N_o + \sum_{k=1}^K \sum_{t' \in T_k} \frac{P}{(1+\|t'-r\|)^\alpha}} \\
&= \frac{\frac{P}{(1+C)^\alpha}}{N_o + \sum_{k=1}^K |T_k| \frac{P}{(1+2k\delta)^\alpha}} < \frac{1}{(1+C)^\alpha} \frac{P}{\sum_{k=1}^{K-1} 7k \frac{P}{(1+2k\delta)^\alpha}} \\
&< \frac{1}{7(1+C)^\alpha} \frac{1}{\sum_{k=1}^{K-1} \frac{k}{(k+2k\delta)^\alpha}} = \frac{1}{7} \left(\frac{1+2\delta}{1+C} \right)^\alpha \frac{1}{\sum_{k=1}^{K-1} \frac{1}{k^{\alpha-1}}} \\
&< \frac{1}{7} \left(\frac{1+2C(2+D)}{1+C} \right)^\alpha \frac{1}{\sum_{k=1}^{\lfloor \sqrt{\frac{m}{7}} - 2 \rfloor} \frac{1}{k^{\alpha-1}}}
\end{aligned}$$

where we used (A1) in the first inequality, (A2) in the third inequality. This proves Part (a). Part (b) follows from Part (a) by choosing m sufficiently large, and Part (c) follows from Part (b) by choosing C, D sufficiently small. \square

Appendix B

Condition (10) implies the connectivity condition of [12]

We show that condition (10), which requires $an\left(\frac{C_n}{n^\gamma}\right)^2 + \ln \frac{C_n}{n^\gamma} \rightarrow \infty$ implies the necessary and sufficient condition for connectivity, namely, $n\left(\frac{C_n}{n^\gamma}\right)^2 - \ln n \rightarrow \infty$. For brevity, let $\rho_n = \frac{C_n}{n^\gamma}$. We begin by using proof by contradiction to show that if (10) holds, i.e. $an\rho_n^2 + \ln \rho_n \rightarrow \infty$, and $1 < b < \sqrt{\frac{1}{a}}$, then there exists n_o such that $\rho_n > b\sqrt{\frac{\ln n}{n}}$ for all $n \geq n_o$. (Recall that $a = \frac{1}{2^{13}\pi} \ln \frac{e}{2} < 1$.) Accordingly, suppose to the contrary that for all n_o there exists $n \geq n_o$ such that $\rho_n \leq b\sqrt{\frac{\ln n}{n}}$. Let us choose n_o large enough that $b\sqrt{\frac{\ln n_o}{n_o}} < 1$. Since $\frac{\rho^2}{|\ln \rho|} = \frac{\rho^2}{-\ln \rho}$ is monotonic increasing for $\rho < 1$, there exists $n \geq n_o$ such that

$$an\rho_n^2 + \ln \rho_n \leq ab^2 \ln n + \ln b \sqrt{\frac{\ln n}{n}} = (ab^2 - 1) \ln n + \frac{1}{2} \ln \ln n + \ln b.$$

Since (1) there are infinitely many n for which the above holds, (2) $(ab^2 - 1) < 0$, and (3) for large n the right hand side above approaches $-\infty$, it follows that $\liminf_{n \rightarrow \infty} an\rho_n^2 + \ln \rho_n = -\infty$, which contradicts the assumption that $n\rho_n^2 + \ln \rho_n \rightarrow \infty$. Therefore, there exists n_o such that $\rho_n > b\sqrt{\frac{\ln n}{n}}$ for all $n \geq n_o$. Since for all such n , $n\rho_n^2 - \ln n > b \ln n - \ln n = (b-1) \ln n$, and since $b > 1$, we conclude that $n\rho_n^2 - \ln n \rightarrow \infty$, which is the desired result.

Appendix C

Lemma C1 *For any $w > 0$ and $\rho \geq 2w$, there exists a partition of a disk of radius ρ into convex cells such that each cell has diameter no larger than $8w$ and area at least $w^2/8$.*

Note that we have not attempted to make the bounds in this lemma as tight as possible.

Proof: Given $w > 0$ and $\rho \geq 2w$, we will specify a set of points G in the disk of radius ρ and show that the Voronoi partition corresponding to this set has the desired properties. We will begin by

choosing $u > 0$ and a positive integer m such that $(m + 1/2)u = \rho$ and $w \leq u < 2w$. Specifically, choose m such that $\rho = (m + 1/2)w + r$, for some r , $0 \leq r < w$, and choose $u = \rho/(m + 1/2)$. Since $\rho > 2w$, it must be that $m \geq 1$. Since $(m + 1/2)w + r = (m + 1/2)u$, we have $w \leq u = w + r/(m + 1/2) < w + w = 2w$.

To specify, G , we first place the center of the disk into G , which we consider to be the origin of the Cartesian plane. Next, for $d = 1, \dots, m$, add points on a ring of radius du to G in such a way that the Euclidean distances from each point to its two immediate neighbors on the ring are at least $u/2$ and no more than u . For $d = 1$, one can simply add to G the vertices of a regular hexagon with sides of length u centered at the origin. For $d \geq 2$, start by placing one point, denoted p_1 , arbitrarily on the ring. Place a second point p_2 on the ring at distance $u/2$ from the first. To place the third point p_3 , move on the ring away from p_1 and p_2 to a point at distance $u/2$ from p_2 . Continue in this way to add points on the ring until the n th point p_n is within distance $u/2$ of p_1 . Discarding p_n , one obtains the set of points $\{p_1, \dots, p_{n-1}\}$. Clearly every point in this set is at distance $u/2$ from both of its immediate neighbors, except possibly for p_1 and p_{n-1} . However, $\|p_{n-1} - p_1\| > u/2$, since p_{n-1} was not the last point picked, and by the triangle inequality $\|p_{n-1} - p_1\| \leq \|p_{n-1} - p_n\| + \|p_n - p_1\| \leq \frac{u}{2} + \frac{u}{2} = u$. Thus, the set $\{p_1, \dots, p_{n-1}\}$ has the desired property that the distances from each point to its immediate neighbors on the ring are at least $u/2$ and no more than u .

Since points on distinct rings are at least u apart, we see that every point is at least $u/2$ apart from every other point. Let Π denote a Voronoi partition for this set. That is, Π is a partition with one cell for each point in G , and with each x in the disk being contained in a cell corresponding to a point in G to which it is closest. The cells of a Voronoi partition are convex.

Consider the Voronoi cell corresponding to some point $p \in G$. For any x in this cell, the distance to p is at most $2u$, because the distance from x to the closest spot (not necessarily a point in G) on the ring containing p is at most u , and because the distance from this spot to p is at most u . Therefore, the diameter of the cell is at most $4u < 8w$. The fact that any two points are at least $u/2$ from each other implies that every Voronoi cell contains a circle of radius $u/4$. Therefore, its area is at least $\pi u^2/16 > u^2/8 \geq w^2/8$. \square

Lemma C2 *For the partition chosen in Step 1, whose cells have diameter z or less, with $z \leq n^\gamma$,*

$$p_{n,i} \leq 6 \frac{z}{n^\gamma}$$

for all i , where $p_{n,i}$ is the probability that the i th cell of the partition is intersected by a random line whose endpoints are independently drawn from the network region with uniform distributions.

Proof: We upper bound $p_{n,i}$ by the probability, denoted $\bar{p}_{n,i}$, of the cell being intersected by a random line after translating the cell so as to maximize this probability. The translated cell will contain the center of the circular network region, which we consider to be the origin of the coordinate axes. In turn, we bound $\bar{p}_{n,i}$ by bounding $\Pr(\text{line intersect} | r, \theta)$, which is the conditional probability of a line intersecting the translated cell given that s , the source end of the line, is at (r, θ) in polar coordinates. For $r \leq 2z$, we use $\Pr(\text{line intersect} | r, \theta) \leq 1$. For $r > 2z$, in which case s cannot lie in the translated cell, $\Pr(\text{line intersect} | r, \theta)$ equals the probability that the destination end of the random line d lies in the shaded region shown. The latter is bounded above by the area

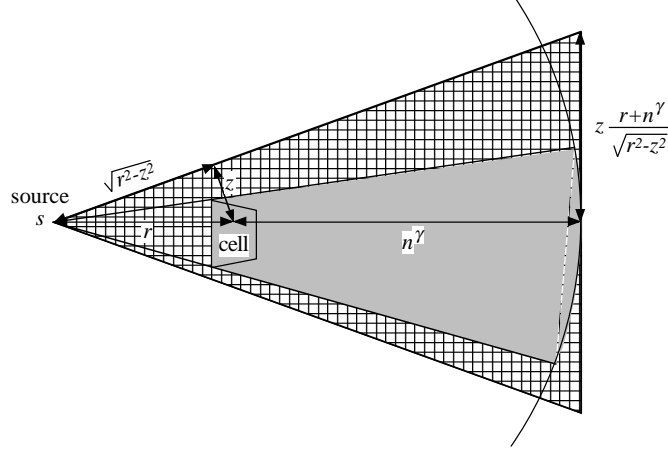


Figure 4: The route from a source s to its destination d passes through the displayed cell if and only if d lies in the shaded region. The probability of this is bounded above by the probability that d lies in the cross-hatched triangle. The cell includes the center of the network region, which has been rotated relative so that s lies horizontally to the left of the origin.

of the crosshatched triangle shown in that figure, divided by the area of the network region. Using the fact that the cell diameter is at most z we find

$$\Pr(\text{line intersect} | r, \theta) \leq \begin{cases} 1, & r \leq 2z \\ \frac{z \frac{r+n^\gamma}{\sqrt{r^2+z^2}} (r+n^\gamma)}{\pi n^{2\gamma}}, & r > 2z \end{cases} < \begin{cases} 1, & r \leq 2z \\ \frac{2z(r+n^\gamma)^2}{\sqrt{3}\pi r n^{2\gamma}}, & r > 2z \end{cases}.$$

Since the probability density of (r, θ) is $p(r, \theta) = \frac{2r}{n^{2\gamma}} \frac{1}{2\pi}$, we have

$$\begin{aligned} p_{n,i} &\leq \bar{p}_{n,i} = \int_0^{2\pi} \int_0^{n^\gamma} \Pr(\text{line intersect} | r, \theta) \frac{2r}{n^{2\gamma}} \frac{1}{2\pi} dr d\theta \\ &< \int_0^{2\pi} \int_0^{2z} 1 \frac{2r}{n^{2\gamma}} \frac{1}{2\pi} dr d\theta + \int_0^{2\pi} \int_{2z}^{n^\gamma} \frac{2z(r+n^\gamma)^2}{\sqrt{3}\pi r n^{2\gamma}} \frac{2r}{n^{2\gamma}} \frac{1}{2\pi} dr d\theta \\ &< 4 \frac{z^2}{n^{2\gamma}} + \frac{32}{3\sqrt{3}\pi} \frac{z}{n^\gamma} < 6 \frac{z}{n^\gamma} \end{aligned}$$

where the next to last inequality uses the fact that $\frac{z}{n^\gamma} < 1$. □

Lemma C3 Let $Y = \sum_{i=1}^n B_i$ be the sum of n independent and identical (IID) binary random variables B_1, \dots, B_n , with $\Pr(B_i = 1) = q = 1 - \Pr(B_i = 0)$ and $0 < q < 1$. Then for any $1 \leq \nu < 1/q$

$$\Pr(Y > \nu nq) \leq \exp \left\{ -nq \left(\nu \ln \frac{\nu}{e} + 1 \right) \right\} \quad (\text{C1})$$

and for any $0 < \nu \leq 1$

$$\Pr(Y < \nu nq) \leq \exp \left\{ -nq \left(\nu \ln \frac{\nu}{e} + 1 \right) \right\}. \quad (\text{C2})$$

Proof: Suppose $0 < q < 1$ and $1 \leq \nu < 1/q$. Using the Chernoff bound and the IID nature of the

B_i 's, we have

$$\begin{aligned}\Pr(Y > \nu nq) &\leq \min_{s \geq 0} E \left[e^{s(Y - \nu nq)} \right] = \min_{s \geq 0} \prod_{i=1}^n E \left[e^{s(B_i - \nu q)} \right] \\ &= \left(\min_{s \geq 0} E \left[e^{s(B_1 - \nu q)} \right] \right)^n = e^{-nD(\nu q \| q)}\end{aligned}\quad (\text{C3})$$

where $D(\nu q \| q)$ denotes the divergence of the probability distribution $\{\nu q, 1 - \nu q\}$ with respect to the distribution $\{q, 1 - q\}$, which is defined and bounded below:

$$\begin{aligned}D(\nu q \| q) &\triangleq \nu q \ln \frac{\nu q}{q} + (1 - \nu q) \ln \frac{1 - \nu q}{1 - q} = \nu q \ln \frac{\nu q}{q} + (1 - q) \frac{1 - \nu q}{1 - q} \ln \frac{1 - \nu q}{1 - q} \\ &\geq \nu q \ln \frac{\nu q}{q} + (1 - q) \left(\frac{1 - \nu q}{1 - q} - 1 \right) = \nu q \ln \nu + q - \nu q = q \left(\nu \ln \frac{\nu}{e} + 1 \right)\end{aligned}$$

where the inequality in the above uses $x \ln x \geq x - 1$. Substituting the above into (C3) gives (C1).

Now suppose $0 < q < 1$ and $0 < \nu \leq 1$. Using the Chernoff bound and the IID nature of the B_i 's, we have

$$\begin{aligned}\Pr(Y < \nu nq) &\leq \min_{s \geq 0} E \left[e^{-s(Y - \nu nq)} \right] = \min_{s \geq 0} \prod_{i=1}^n E \left[e^{-s(B_i - \nu q)} \right] \\ &= \left(\min_{s \geq 0} E \left[e^{-s(B_1 - \nu q)} \right] \right)^n = e^{-nD(\nu q \| q)}.\end{aligned}\quad (\text{C4})$$

From here, the proof is the same as for the previous case. \square

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